The strikingly good approximation was computed over and over by Gauss at intervals up to 3 million, all computed by Gauss himself who could determine the number of primes in a chiliad (block of one thousand numbers) in 15 minutes.

Riemann, in 1859, in a paper [R] written on the occasion of his admission to the Berlin Academy of Sciences and read to the Academy by none other than Encke, devised an analytic way to understand the error term in Gauss' approximation, via the zeros of the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The connection of $\zeta(s)$ with prime numbers was found by Euler via his product formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

where there is one factor for each prime number p. This formula encodes the fundamental theorem of arithmetic that every integer is a product of primes in a unique way. Riemann saw that the zeros of what we now call the Riemann zeta-function were the key to an analytic expression for $\pi(x)$. Riemann observes that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^{\infty} \psi(x) x^{s-1}\, dx$$

where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

and then uses

$$2\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right),$$

which follows from a formula of Jacobi, to transform the part of the integral on $0 \le x \le 1$. In this way he finds that $\zeta(s)$ is a meromorphic function of s with its only a pole a simple pole at $s = 1$ and that

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is an entire function of order 1 which satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

As a consequence, $\zeta(s)$ has zeros at $s = -2n$ for $n \in \mathbb{Z}_+$, these are the so called trivial zeros, as well as a denser infinite sequence of zeros in the "critical strip"

$$0 \le \Re s \le 1.$$

The Euler product precludes any zeros with real part larger than 1. Also, for any non-trivial zero $\rho = \beta + i\gamma$ there is a dual zero $1-\rho$ by the functional equation. Also, $\overline{\rho}$ and $1-\overline{\rho}$ are zeros since $\zeta(s)$ is real for real s but note that $\overline{\rho}$ coincides with $1-\rho$ whenever $\beta = 1/2$. Riemann used Stirling's formula and the functional equation to evaluate the number of non-trivial zeros in the critical strip as

$$N(T) := \#\{\rho = \beta + i\gamma : 0 < \gamma \le T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + 7/8 + S(T) + O(1/T)$$

where

$$S(T) = \frac{1}{\pi} \arg \zeta(1/2 + iT)$$

where the argument is determined by beginning with $\arg \zeta(2) = 1$ and continuous variation along line segments from 2 to $2 + iT$ and then to $1/2 + iT$, taking appropriate action if a zero is on the path. Riemann implied that $S(T) = O(\log T)$ a fact that was later proven rigorously by Backlund [Bac18]. Thus, the zeros get denser as one moves up the critical strip.

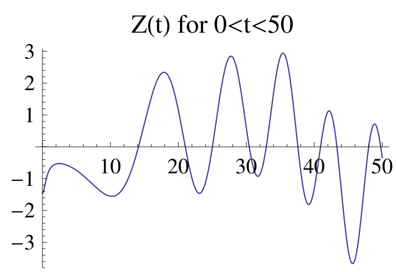
The functional equation together with Riemann's formula for the number of zeros of $\zeta(s)$ up to a height T help give us a picture of $\zeta(1/2 + it)$. In particular Hardy defined a function $Z(t)$ which is a real function of a real variable having the property that $|\zeta(1/2+it)| = |Z(t)|$. It may be defined by

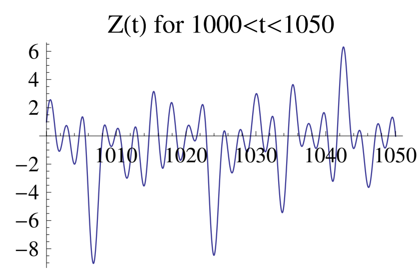
$$Z(t) = \chi(1/2 - it)^{1/2} \zeta(1/2 + it)$$

where $\chi(s)$ is the factor from the functional equation which may be written in asymmetric form as

$$\chi(1 - s) = 2(2\pi)^{-s} \gamma(s) \cos \frac{\pi s}{2}.$$

Here are some graphs of $Z(t)$:





1.1. **Riemann's formula for primes.** Riemann found an exact formula for $\pi(x)$. If we invert Euler's formula we find

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = 1 - 2^{-s} - 3^{-s} - 5^{-s} + 6^{-s} + \cdots = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where μ is known as the Möbius mu-function. A simple way to explain the value of $\mu(n)$ is that it is 0 if n is divisible by the square of any prime, while if n is squarefree then it is $+1$ if n has an even number of prime divisors and -1 if n has an odd number of prime divisors. Riemann's formula is

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n})$$